The contingent epiderivative and the calculus of variations on time scales*

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Abstract

The calculus of variations on time scales is considered. We propose a new approach to the subject that consists in applying a differentiation tool called the *contingent epiderivative*. It is shown that the contingent epiderivative applied to the calculus of variations on time scales is very useful: it allows to unify the delta and nabla approaches previously considered in the literature. Generalized versions of the Euler–Lagrange necessary optimality conditions are obtained, both for the basic problem of the calculus of variations and isoperimetric problems. As particular cases one gets the recent delta and nabla results.

Keywords: contingent derivative; contingent epiderivative; calculus of variations; Euler–Lagrange equations; time scales; delta and nabla calculi.

2010 Mathematics Subject Classification: 49K05; 26E70; 34N05.

1 Introduction

The *contingent epiderivative* was introduced by J.P. Aubin (see [5]) in order to differentiate real valued functions and, in general, functions with extended

*Submitted: 06/March/2010; revised: 12/May/2010; accepted: 03/July/2010; for publication in *Optimization—A Journal of Mathematical Programming and Operations Research*.

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values (functions with values in $\mathbb{R} \cup \{\pm\infty\}$). This derivative belongs to the wide spectrum of generalized differentiation tools, which are applied in optimal control theory. For applications of the contingent epiderivative to optimality conditions one can refer to [8, 20, 21, 30, 31] and references therein. In order to define the contingent epiderivative one has to be familiar with set-valued maps (maps that have sets as their values): the definition of the contingent epiderivative is based on the *contingent derivative* [4,5] (also called *graphical derivative*) that is a set-valued map that "differentiates" set-valued maps. Namely, we associate with a function f the set-valued map $F_\uparrow$ defined by $F_\uparrow(x) := \langle f(x), +\infty)$, whose graph is the epigraph of f. The graphs of the derivatives of such set-valued maps $F_\uparrow$ are the epigraphs of functions which are called *epiderivatives*. The contingent derivative was also introduced and defined by Aubin in [5]. In this paper we use such generalized differentiation tools in order to unify previous results on the calculus of variations on time scales.

The mathematics of *time scales* was introduced by Aulbach and Hilger as a tool to unify and extend the theories of difference and differential equations [1,6,19]. A time scale is a model of time, and the new theory has found important applications in several fields that require simultaneous modeling of discrete and continuous data, in particular in the calculus of variations — see [2, 3, 7, 9, 10, 14–16, 23–26, 28, 29] and references therein. Looking to the above papers one concludes that in the calculus of variations on time scales the problems are formulated and the results are proved in terms of the delta or the nabla calculus. Here we propose a more general approach: we make use of contingent epiderivatives to unify the two different approaches followed in the literature.

Let $\mathbb{T}$ be a given time scale with jump operators σ and ρ, and time scale derivatives Δ and ∇. We consider as the basic problem of the calculus of variations to minimize or maximize

$$\mathcal{L}[y] = \int_a^b L(t, (y \circ \xi_u)(t), D_\uparrow \overline{y}(t)(u)) d_u t$$

subject to boundary conditions $y(a) = \alpha$ and $y(b) = \beta$, where $D_\uparrow$ denotes the contingent epiderivative,

$$d_u t := \begin{cases} u\Delta t, & \text{if } u \geq 0 \\ u\nabla t, & \text{if } u \leq 0 \,, \end{cases}$$

and

$$y \circ \xi_u := \begin{cases} u(y \circ \sigma), & \text{if } u \geq 0 \\ u(y \circ \rho), & \text{if } u \leq 0. \end{cases}$$

In the particular case $u = 1$ we obtain the delta variational problem studied in [9]; when $u = -1$ we obtain the nabla variational problem studied in [3] (see also [28]). Moreover, we consider isoperimetric problems of the calculus of variations on time scales: to minimize or maximize $\mathcal{L}[y]$ subject to boundary conditions $y(a) = \alpha$, $y(b) = \beta$, and an additional constraint of the form

$$\mathcal{K}[y] = \int_a^b G(t, (y \circ \xi_w)(t), D_\uparrow \overline{y}(t)(w)) d_w t = K \,.$$

As particular cases we obtain the problem investigated in [16] by choosing $u = w = 1$; we obtain the problem investigated in [2] by choosing $u = w = -1$.

Main results of the paper give necessary optimality conditions of Euler–Lagrange type for the basic and isoperimetric problems that we propose in this paper. As direct corollaries we obtain the previous delta and nabla results found in [2, 3, 9, 16].

The paper is organized as follows. In Section 2 some preliminary results on time scales are presented. Our results are then given in Section 3. Finally, we end with Section 4 of main conclusions and some perspectives on future directions.

2 Preliminaries

In this section we introduce the basic definitions and results that will be needed in the sequel. For a more general presentation of the theory of time scales and detailed proofs, we refer the reader to the books [11, 12, 22].

2.1 The delta calculus

A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$. Besides standard cases of $\mathbb{R}$ (continuous time) and $\mathbb{Z}$ (discrete time), many different models of time may be used, e.g., the h-numbers ($\mathbb{T} = h\mathbb{Z} := \{hz \mid z \in \mathbb{Z}\}$, where $h > 0$ is a fixed real number) and the q-numbers ($\mathbb{T} = q^{\mathbb{N}_0} := \{q^k \mid k \in \mathbb{N}_0\}$, where $q > 1$ is a fixed real number). We assume that a time scale $\mathbb{T}$ has the topology that it inherits from the real numbers with the standard topology.

For each time scale $\mathbb{T}$ the following operators are used:

- the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$, defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ for $t \neq \sup \mathbb{T}$ and $\sigma(\sup \mathbb{T}) = \sup \mathbb{T}$ if $\sup \mathbb{T} < +\infty$;
- the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$, defined by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$ for $t \neq \inf \mathbb{T}$ and $\rho(\inf \mathbb{T}) = \inf \mathbb{T}$ if $\inf \mathbb{T} > -\infty$;
- the *forward graininess function* $\mu : \mathbb{T} \to [0, \infty)$, defined by $\mu(t) := \sigma(t) - t$;
- the *backward graininess function* $\nu : \mathbb{T} \to [0, \infty)$, defined by $\nu(t) = t - \rho(t)$.

A point $t \in \mathbb{T}$ is called *right-dense*, *right-scattered*, *left-dense* or *left-scattered* if $\sigma(t) = t$, $\sigma(t) > t$, $\rho(t) = t$, and $\rho(t) < t$, respectively. We say that t is *isolated* if $\rho(t) < t < \sigma(t)$, that t is *dense* if $\rho(t) = t = \sigma(t)$. If $\sup \mathbb{T}$ is finite and left-scattered, we define $\mathbb{T}^\kappa := \mathbb{T} \setminus \{\sup \mathbb{T}\}$, otherwise $\mathbb{T}^\kappa := \mathbb{T}$.

Definition 1. *We say that a function $f : \mathbb{T} \to \mathbb{R}$ is* delta differentiable *at $t \in \mathbb{T}^\kappa$ if there exists a number $f^\Delta(t)$ such that for all $\varepsilon > 0$ there is a neighborhood U of t (i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that*

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s|, \text{ for all } s \in U.$$

We call $f^{\Delta}(t)$ the delta derivative *of f at t and f is said to be* delta differentiable *on $\mathbb{T}^{\kappa}$ provided $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}^{\kappa}$.*

Remark 2. *If $t \in \mathbb{T} \setminus \mathbb{T}^{\kappa}$, then $f^{\Delta}(t)$ is not uniquely defined, since for such a point t, small neighborhoods U of t consist only of t and, besides, we have $\sigma(t) = t$. For this reason, maximal left-scattered points are omitted in Definition 1.*

Note that in right-dense points $f^{\Delta}(t) = \lim_{s \to t} \frac{f(t)-f(s)}{t-s}$, provided this limit exists, and in right-scattered points $f^{\Delta}(t) = \frac{f(\sigma(t))-f(t)}{\mu(t)}$, provided f is continuous at t.

For $f : \mathbb{T} \to X$, where X is an arbitrary set, we define $f^{\sigma} := f \circ \sigma$. For delta differentiable f and g, the next formulas hold:

$$\begin{aligned} f^{\sigma}(t) &= f(t) + \mu(t) f^{\Delta}(t)\,, \\ (fg)^{\Delta}(t) &= f^{\Delta}(t) g^{\sigma}(t) + f(t) g^{\Delta}(t) \\ &= f^{\Delta}(t) g(t) + f^{\sigma}(t) g^{\Delta}(t). \end{aligned}$$

In order to describe a class of functions that possess a delta antiderivative, the following definition is introduced:

Definition 3. *A function $f : \mathbb{T} \to \mathbb{R}$ is called* regulated *provided its right-sided limits exists (finite) at all right-dense points in $\mathbb{T}$ and its left-sided limits exists (finite) at all left-dense points in $\mathbb{T}$.*

Definition 4. *A function $f : \mathbb{T} \to \mathbb{R}$ is called* rd-continuous *if it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$.*

Definition 5. *A function $F : \mathbb{T} \to \mathbb{R}$ is called a* delta antiderivative *of $f : \mathbb{T} \to \mathbb{R}$ provided $F^{\Delta}(t) = f(t)$ for all $t \in \mathbb{T}^{\kappa}$. In this case we define the* delta integral *of f from a to b ($a, b \in \mathbb{T}$) by*

$$\int_a^b f(t) \Delta t := F(b) - F(a)\,.$$

Theorem 6. *(Theorem 1.74 in [11]) Every rd-continuous function has a delta antiderivative. In particular, if $a \in \mathbb{T}$, then the function F defined by*

$$F(t) = \int_a^t f(\tau) \Delta\tau, \quad t \in \mathbb{T}\,,$$

is a delta antiderivative of f.

2.2 The nabla approach

In order to introduce the definition of nabla derivative, we define a new set $\mathbb{T}_{\kappa}$ which is derived from $\mathbb{T}$ as follows: if $\mathbb{T}$ has a right-scattered minimum m,

then $\mathbb{T}_\kappa = \mathbb{T} \setminus \{m\}$; otherwise, $\mathbb{T}_\kappa = \mathbb{T}$. In order to simplify expressions, and similarly as done with composition with σ, we define $f^\rho(t) := f(\rho(t))$. We say that a function $f : \mathbb{T} \to \mathbb{R}$ is *nabla differentiable* at $t \in \mathbb{T}_\kappa$ if there is a number $f^\nabla(t)$ such that for all $\varepsilon > 0$ there exists a neighborhood U of t (i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$|f^\rho(t) - f(s) - f^\nabla(t)(\rho(t) - s)| \leq \varepsilon|\rho(t) - s|, \text{ for all } s \in U.$$

We call $f^\nabla(t)$ the *nabla derivative* of f at t. Moreover, we say that f is *nabla differentiable* on $\mathbb{T}$ provided $f^\nabla(t)$ exists for all $t \in \mathbb{T}_\kappa$. If f is nabla differentiable at t, then

$$f^\rho(t) = f(t) - \nu(t) f^\nabla(t)\,.$$

Theorem 7. *(Theorem 8.41 in [11]) Suppose $f, g : \mathbb{T} \to \mathbb{R}$ are nabla differentiable at $t \in \mathbb{T}_\kappa$. Then,*

1. *the sum $f + g : \mathbb{T} \to \mathbb{R}$ is nabla differentiable at t and*

$$(f+g)^\nabla(t) = f^\nabla(t) + g^\nabla(t)\,;$$

2. *for any constant α, $\alpha f : \mathbb{T} \to \mathbb{R}$ is nabla differentiable at t and*

$$(\alpha f)^\nabla(t) = \alpha f^\nabla(t)\,;$$

3. *the product $fg : \mathbb{T} \to \mathbb{R}$ is nabla differentiable at t and*

$$\begin{aligned} (fg)^\nabla(t) &= f^\nabla(t) g(t) + f^\rho(t) g^\nabla(t) \\ &= f^\nabla(t) g^\rho(t) + f(t) g^\nabla(t). \end{aligned}$$

Definition 8. *A function $F : \mathbb{T} \to \mathbb{R}$ is called a* nabla antiderivative *of $f : \mathbb{T} \to \mathbb{R}$ provided $F^\nabla(t) = f(t)$ for all $t \in \mathbb{T}_\kappa$. In this case we define the* nabla integral *of f from a to b ($a, b \in \mathbb{T}$) by*

$$\int_a^b f(t) \nabla t := F(b) - F(a)\,.$$

In order to exhibit a class of functions that possess a nabla antiderivative, the following definition is introduced.

Definition 9. *Let $\mathbb{T}$ be a time scale, $f : \mathbb{T} \to \mathbb{R}$. We say that function f is* ld-continuous *if it is continuous at left-dense points and its right-sided limits exist (finite) at all right-dense points.*

Theorem 10. *(Theorem 8.45 in [11]) Every ld-continuous function has a nabla antiderivative. In particular, if $a \in \mathbb{T}$, then the function F defined by*

$$F(t) = \int_a^t f(\tau) \nabla \tau, \quad t \in \mathbb{T}\,,$$

is a nabla antiderivative of f.

For more on the nabla calculus we refer the reader to [12, Chap. 3].

3 Main Results

In this section, which is the main part of the paper, we present a unified Euler–Lagrange equation and unified necessary optimality conditions for isoperimetric problems on time scales. The differentiation tool we use for such unification is the contingent epiderivative. We consider $\mathbb{T}$ to be a given time scale with $\inf \mathbb{T} := a$, $\sup \mathbb{T} := b$, and $I := [a, b] \cap \mathbb{T}$ for $[a, b] \subset \mathbb{R}$. Moreover, by I_κ^κ (or $\mathbb{T}_\kappa^\kappa$) we mean $I_\kappa^\kappa := I^\kappa \cap I_\kappa$ (respectively, $\mathbb{T}_\kappa^\kappa := \mathbb{T}^\kappa \cap \mathbb{T}_\kappa$).

3.1 Contingent epiderivatives

A *set-valued map* (*multifunction*) $F : \mathbb{R} \to 2^{\mathbb{R}}$ is a map that has sets as its values. We denote it by $F : \mathbb{R} \twoheadrightarrow \mathbb{R}$. Every set-valued map $F : \mathbb{R} \twoheadrightarrow \mathbb{R}$ is characterized by its *graph*, Graph(F), as the subset of $\mathbb{R}^2$ defined by

$$\mathrm{Graph}(F) := \left\{(x, y) \in \mathbb{R}^2 : y \in F(x)\right\}.$$

We shall say that $F(x)$ is the *image* or the *value* of F at x. A set-valued map is said to be *nontrivial* if its graph is not empty, i.e., if there exists an element $x \in \mathbb{R}$ such that $F(x)$ is not empty. The *domain* of F is the set of elements $x \in \mathbb{R}$ such that $F(x)$ is not empty: $\mathrm{Dom}(F) = \{x \in \mathbb{R} : F(x) \neq \emptyset\}$. The *image* of F is the union of the images (or values) $F(x)$, when x ranges over $\mathbb{R}$: $\mathrm{Im}(F) = \bigcup_{x\in\mathbb{R}} F(x)$.

Definition 11. *Let X be any nonempty subset of $\mathbb{R}$. By* the epigraph of $f : X \to \mathbb{R}$, *denoted by Epi(f), we mean the following set:*

$$Epi(f) := \left\{(t, \lambda) \in X \times \mathbb{R} : f(t) \leq \lambda\right\}.$$

The hypograph of $g : X \to \mathbb{R}$ *is defined in the symmetric way: $Hyp(g) := \{(t, \lambda) \in X \times \mathbb{R} : g(t) \geq \lambda\}$.*

Remark 12. *Note that $Epi(f) \cap Hyp(f) = Graph(f)$.*

If $X = \mathbb{T}$ is a time scale, then we can use the same definition of epigraph by introducing an appropriate extension of the epigraph of a function $f : \mathbb{T} \to \mathbb{R}$. Let $G(f)$ denote the following set:

$$G(f) = \bigcup_{t\in\mathbb{T}} \left\{\alpha(t, y) + \beta(\sigma(t), z) : \, y \geq f(t), z \geq f^\sigma(t)\right\},$$

where $\alpha + \beta = 1, \alpha, \beta \geq 0$.

Remark 13. *It is easy to see that $G(f) \subset \overline{conv} Epi(f)$, where $\overline{conv} Epi(f)$ is the closure of the convex hull of $Epi(f)$.*

Using the formulation of $G(f)$ we can assign to $f : I \to \mathbb{R}$ the function $\overline{f} : [a, b] \to \mathbb{R}$ defined by the formula

$$\mathrm{Epi}(\overline{f}) = G(f). \tag{1}$$

Let us notice that for $f, g : I \to \mathbb{R}$ and $a, b \in \mathbb{R}$ it holds: $a\overline{f} + b\overline{g} = \overline{af + bg}$.

Remark 14. *Note that function $\overline{f}$ defined by formula* (1) *can be presented in the following way (see, e.g., [13]):*

$$\overline{f}(t) = \begin{cases} f(t), & \text{if } t \in \mathbb{T} \\ f(s) + \frac{f(\sigma(s)) - f(s)}{\mu(s)}(t-s), & \text{if } t \in (s, \sigma(s)), \end{cases}$$

where $s \in \mathbb{T}$ and s is right-scattered or

$$\overline{f}(t) = \begin{cases} f(t), & \text{if } t \in \mathbb{T} \\ f(s) + \frac{f(s) - f(\rho(s))}{\nu(s)}(t-s), & \text{if } t \in (\rho(s), s), \end{cases}$$

where $s \in \mathbb{T}$ and s is left-scattered.

We recall a general definition of the contingent cone to a subset of $\mathbb{R}^2$. This theory is presented for a subset of any normed space X in [5].

Definition 15 ([5])**.** *Let $K \subset \mathbb{R}^2$ and point $p \in \overline{K}$ belong to the closure of K. The contingent cone $T_K(p)$ is defined by*

$$T_K(p) := \{v : \liminf_{h \to 0+} d(p + hv, K)/h = 0\},$$

where $d(q, K) = \inf_{k \in K} d(q, k)$ is the distance between point $q = p + hv$ and set K.

Remark 16 ([5])**.** *$T_K(p)$ is a closed set.*

The following characterization of the contingent cone in terms of sequences is very convenient: $v \in T_K(p)$ if and only if there exists $h_n \to 0^+$ and $v_n \to v$ such that for all n, $p + h_n v_n \in K$.

Remark 17 ([5])**.** *If $p \in Int(K)$, then $T_K(p) = \mathbb{R}^2$.*

Definition 18 ([5])**.** *Let $F : \mathbb{R} \twoheadrightarrow \mathbb{R}$ be a set-valued map. The* contingent derivative of F at $p \in \text{Graph}(F)$*, denoted by $DF(p)$, is the set-valued map from $\mathbb{R}$ to $\mathbb{R}$ defined by*

$$Graph\,(DF(p)) := T_{Graph(F)}(p)\,.$$

For $F := f$ a single-valued function, we set

$$Df(x) := Df(x, f(x)).$$

Let us point out that

$$\text{Graph}\,(DF(p)) = \{u = (u_1, u_2) : u_2 \in DF(p)(u_1)\}$$

and $u \in T_{Graph(F)}(p)$. In [5] one can find the following characterization of the contingent derivative:

Proposition 19 (cf. [5], Proposition 5.1.4, p.186)**.** *Let $F : \mathbb{R} \twoheadrightarrow \mathbb{R}$ be a set-valued map and let $p = (x, y) \in Graph(F)$. Then,*

$$v \in DF(p)(u) \Leftrightarrow \liminf_{h \to 0+, u' \to u} d\left(v, \frac{F(x + hu') - y}{h}\right) = 0\,.$$

If $p \in Int(Dom(F))$ and F is Lipschitz around p, then

$$v \in DF(p)(u) \Leftrightarrow \liminf_{h \to 0+} d\left(v, \frac{F(x + hu) - y}{h}\right) = 0\,.$$

Let $X \subset \mathbb{R}$, possibly a time scale, and let us consider an extended function $f : X \to \mathbb{R} \cup \{\pm\infty\}$ whose domain is $\mathrm{Dom}(f) = \{t \in X : f(t) \neq \pm\infty\}$. We call an extended function *nontrivial* if $\mathrm{Dom}(f) \neq \emptyset$. To introduce elements of the theory of contingent epiderivatives for functions on time scales we define two set-valued maps $F_\uparrow : X \twoheadrightarrow \mathbb{R}$ and $F_\downarrow : X \twoheadrightarrow \mathbb{R}$, both corresponding to $f : X \to \mathbb{R}$, in the following way (see [5]):

Definition 20. *Let $X \subset \mathbb{R}$ and $f : X \to \mathbb{R} \cup \{\pm\infty\}$ be an extended function with nonempty domain $Dom(f) \neq \emptyset$. Then*

a) $F_\uparrow(t) = \begin{cases} f(t) + \mathbb{R}_+ & \text{if } t \in Dom(f) \\ \emptyset & \text{if } f(t) = +\infty \\ \mathbb{R} & \text{if } f(t) = -\infty \end{cases}$,

b) $F_\downarrow(t) = \begin{cases} f(t) - \mathbb{R}_+ & \text{if } t \in Dom(f) \\ \emptyset & \text{if } f(t) = -\infty \\ \mathbb{R} & \text{if } f(t) = +\infty \end{cases}$,

where $f(t) + \mathbb{R}_+ = \{\lambda : f(t) \leq \lambda\}$ and $f(t) - \mathbb{R}_+ = \{\lambda : f(t) \geq \lambda\}$.

Remark 21. *It is easy to prove that $Graph(F_\uparrow) = Epi(f) = \bigcup_t \{(t, y) : y \in F_\uparrow(t)\}$ and $Graph(F_\downarrow) = Hyp(f)$.*

With set-valued functions $F_\uparrow$ and $F_\downarrow$ one can associate their contingent derivatives. We naturally have that values of the contingent derivative of $F_\uparrow : \mathbb{R} \twoheadrightarrow \mathbb{R}$ are half lines in the sense that $\forall\ \lambda \geq f(t)$, $\forall\ u \in \mathrm{Dom}\,(DF_\uparrow(t, \lambda))$, it holds:

$$DF_\uparrow(t, \lambda)(u) = DF_\uparrow(t, \lambda)(u) + \mathbb{R}_+\,.$$

Definition 22 ([5])**.** *Let $f : X \to \mathbb{R} \cup \{\pm\infty\}$ be a function with extended values and nonempty domain and let $t \in Dom(f)$. We say that the extended function $D_\uparrow f(t) : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ defined by*

$$D_\uparrow f(t)(u) := \inf\{v : v \in DF_\uparrow(t, f(t))(u)\}\,,$$

where $F_\uparrow$ is given by Definition 20 and $u \in \mathbb{R}$, is the contingent epiderivative of f at t in the direction u. *The function f is said to be* contingently epidifferentiable at t *if its contingent epiderivative never takes the value $-\infty$. If $DF_\uparrow(t, f(t))(u) = \emptyset$ then we set $D_\uparrow f(t)(u) = +\infty$.*

Remark 23 ([5])**.** *Notice that for a function $f : X \to \mathbb{R} \cup \{\pm\infty\}$ with nonempty domain the following conditions are equivalent:*

(i) f *is contingently epidifferentiable at* $t \in Dom(f)$*;*

(ii) $D_{\uparrow} f(t)(0) = 0$.

In [4] one can find the following characterization of the contingent epiderivative for some particular situations:
a) $DF_{\uparrow}(t, f(t))(u) = \mathbb{R}$ iff $D_{\uparrow} f(t)(u) = -\infty$;
b) $DF_{\uparrow}(t, f(t))(u) = [v_0, +\infty)$ iff $D_{\uparrow} f(t)(u) = v_0$.
Now let associate with $f : I \to \mathbb{R}$ the function $\overline{f} : [a, b] \to \mathbb{R} \cup \{\pm\infty\}$ defined by formula (1). Let $\overline{F}_{\uparrow}$ denote the corresponding multifunction for $\overline{f}$ according to Definition 20.

Proposition 24. *For all* $p \in Graph\,(F_{\uparrow})$ *and for each* $u \in \mathbb{R}$ *we have*

$$DF_{\uparrow}(p)(u) \subset D\overline{F}_{\uparrow}(p)(u)\,. \tag{2}$$

Proof. Since $\mathrm{Graph}(F_{\uparrow}) \subset \mathrm{Graph}(\overline{F}_{\uparrow})$ and $\mathrm{Graph}\,(DF_{\uparrow}(p)) = T_{\mathrm{Graph}(F_{\uparrow})}(p)$, it follows that $\mathrm{Graph}(DF_{\uparrow}(p)) \subset \mathrm{Graph}(D\overline{F}_{\uparrow}(p))$. Let $v \in DF_{\uparrow}(p)(u)$. Then $(u, v) \in \mathrm{Graph}\,(DF_{\uparrow}(p)) \subset \mathrm{Graph}\,\big(D\overline{F}_{\uparrow}(p)\big)$ and we get that $v \in D\overline{F}_{\uparrow}(p)(u)$. □

Remark 25. *Inclusion (2) is also satisfied when both sets are empty. It means then that the contingent epiderivative* $D_{\uparrow} f(t)(u) = +\infty$*.*

Remark 26. *Directly from definitions of* $\overline{F}_{\uparrow}$ *and its contingent derivative we have the following:*

1. $DF_{\uparrow}(t)(u) = D\overline{F}_{\uparrow}(t)(u)$ *if* $t \in \mathbb{T}$ *is right-dense and* $u \geq 0$*;*

2. $DF_{\uparrow}(t)(u) = D\overline{F}_{\uparrow}(t)(u)$ *if* $t \in \mathbb{T}$ *is left-dense and* $u \leq 0$*;*

3. $D\overline{F}_{\uparrow}(t)(u) = \emptyset$ *if* $t \in \mathbb{T}$ *is isolated and* $u \neq 0$.

We recall the following useful proposition from [5] showing that the contingent epiderivative can be characterized as a limit of differential quotients:

Proposition 27 ([5])**.** *Let* $g : I \to \mathbb{R} \cup \{\pm\infty\}$ *be a nontrivial extended function and* t *belong to its domain. Then*

$$D_{\uparrow} g(t)(u) = \liminf_{h \to 0+, u' \to u} \frac{g(t + hu') - g(t)}{h}.$$

Corollary 28. *Let* $f : \mathbb{T} \to \mathbb{R} \cup \{\pm\infty\}$ *and* $t \in Dom(f)$*. Then for* $u \in \mathbb{R}$

$$D_{\uparrow} f(t)(u) \geq \liminf_{h \to 0+, u' \to u} \frac{\overline{f}(t + hu') - \overline{f}(t)}{h} = D_{\uparrow} \overline{f}(t)(u)\,.$$

Proof. From Proposition 24 and definition 22 we have $D_{\uparrow} f(t)(u) \geq D_{\uparrow} \overline{f}(t)(u)$. The equality follows from Proposition 27. □

We can state the following relations between delta and nabla derivatives of f at point t (if they exist) and the contingent epiderivative of the corresponding function $\overline{f}$ defined by formula (1).

Proposition 29 ([18]). *Let $t \in \mathbb{T}^{\kappa}_{\kappa}$ and $f : I \to \mathbb{R} \cup \{\pm\infty\}$.*
a) If $f^{\Delta}(t)$ exists, then $D_{\uparrow}\overline{f}(t)(u) = uf^{\Delta}(t)$ for $u \geq 0$.
b) If $f^{\nabla}(t)$ exists, then $D_{\uparrow}\overline{f}(t)(u) = uf^{\nabla}(t)$ for $u \leq 0$.

We can characterize values of the contingent epiderivative of $f : \mathbb{T} \to \mathbb{R} \cup \{\pm\infty\}$ by the contingent epiderivative of $\overline{f}$. Directly from definitions of $\overline{f}$ and its contingent epiderivative we have the following:

Proposition 30 ([18]). *Let $f : I \to \mathbb{R} \cup \{\pm\infty\}$, $\overline{f} : [a, b] \to \mathbb{R} \cup \{\pm\infty\}$, and $Epi(\overline{f}) = G(f)$. Then,*

1. *f is contingently epidifferentiable if and only if $\overline{f}$ is contingently epidifferentiable;*
2. *$D_{\uparrow}f(t)(u) = D_{\uparrow}\overline{f}(t)(u)$ for $t \in \mathbb{T}$ right-dense and $u \geq 0$;*
3. *$D_{\uparrow}f(t)(u) = D_{\uparrow}\overline{f}(t)(u)$ for $t \in \mathbb{T}$ left-dense and $u \leq 0$;*
4. *$D_{\uparrow}\overline{f}(t)(u) = \mathbb{R}_{+}$ for $t \in \mathbb{T}$ isolated and $u \neq 0$.*

We introduce now the following notations and definitions:

Definition 31. *Let $u \in \mathbb{R}$ be a real number. Then,*

$$d_u t := \begin{cases} u\Delta t, & \text{if } u \geq 0 \\ u\nabla t, & \text{if } u \leq 0\,, \end{cases}$$

$$y \circ \xi_u := \begin{cases} u(y \circ \sigma), & \text{if } u \geq 0 \\ u(y \circ \rho), & \text{if } u \leq 0\,. \end{cases}$$

Remark 32. *With the notation of Definition 31 we have*

$$\int_a^b \overline{f}(t) d_u t := \begin{cases} u\int_a^b f(t)\Delta t, & \text{if } u \geq 0 \\ u\int_a^b f(t)\nabla t, & \text{if } u \leq 0\,, \end{cases}$$

where $\overline{f}$ is defined by formula (1).

3.2 The unified Euler–Lagrange equation

Given a time scale $\mathbb{T}$ and $u \in \mathbb{R} \setminus \{0\}$, we consider the question of finding y that is a solution to the problem

$$\begin{gathered} \text{extremize} \quad \mathcal{L}[y] = \int_a^b L(t, (y \circ \xi_u)(t), D_{\uparrow}\overline{y}(t)(u)) d_u t\,, \\ y(a) = \alpha, \quad y(b) = \beta\,, \end{gathered} \tag{3}$$

where function $(t, y, v) \to L(t, y, v)$ from $I \times \mathbb{R}^2$ to $\mathbb{R}$ has partial continuous derivatives with respect to y, v for all $t \in I$.

Let $\partial_i L$ denote the standard partial derivative of $L(\cdot, \cdot, \cdot)$ with respect to its ith variable, $i = 1, 2, 3$.

Any regulated function $y : I \to \mathbb{R}$ such that $D_{\uparrow}\overline{y}(t)(u) \neq \{-\infty, +\infty\}$ is said to be an *admissible* function provided that it satisfies boundary conditions $y(a) = \alpha$, $y(b) = \beta$.

Remark 33. *Proposition 29 implies the following: if y is Δ-differentiable, then for $u = 1$ (3) is just a delta problem of the calculus of variations on time scales (see [7, 9]); while if y is ∇-differentiable, then for $u = -1$ (3) reduces to a nabla problem of the calculus of variations (see [3, 28]). For $u = 0$ the cost-functional to extremize reduces to a constant, so problem (3) is trivial: there is nothing to minimize or maximize; any function satisfying $y(a) = \alpha$ and $y(b) = \beta$ is a solution to (3). Thus, we are interested in the nontrivial case $u \neq 0$.*

Theorem 34. (The unified Euler–Lagrange equation on time scales). *If y is a minimizer or maximizer to problem (3), then y satisfies the following equation:*

$$D_{\uparrow}\left(\partial_3 L\left(t, (y \circ \xi_u)(t), D_{\uparrow}\overline{y}(t)(u)\right)\right)(u) = u \cdot \partial_2 L(t, (y \circ \xi_u)(t), D_{\uparrow}\overline{y}(t)(u)) \quad (4)$$

for all $t \in I_{\kappa^2}^{\kappa^2}$, where $\overline{y}$ is defined by formula (1).

Proof. We consider two cases: $u > 0$ and $u < 0$. For $u > 0$ problem (3) reduces to

$$\begin{gathered} \text{extremize} \quad \int_a^b uL(t, u(y \circ \sigma)(t), uy^{\Delta}(t))\Delta t, \\ y(a) = \alpha, \;\; y(b) = \beta. \end{gathered} \quad (5)$$

If we set $f(t, y^{\sigma}(t), y^{\Delta}(t)) := uL(t, uy^{\sigma}(t), uy^{\Delta}(t))$, then problem (5) is equivalent to

$$\begin{gathered} \text{extremize} \quad \int_a^b f(t, y^{\sigma}(t), y^{\Delta}(t))\Delta t \\ y(a) = \alpha, \;\; y(b) = \beta. \end{gathered} \quad (6)$$

But this is exactly the case of a standard delta problem of the calculus of variations on time scales. For problem (6) the Euler–Lagrange equation of [9] gives the delta equation

$$\left[\partial_3 f(t, y^{\sigma}(t), y^{\Delta}(t))\right]^{\Delta} = \partial_2 f(t, y^{\sigma}(t), y^{\Delta}(t)),$$

which is equivalent to

$$u\left[\partial_3 L(t, u(y \circ \sigma)(t), uy^{\Delta}(t))\right]^{\Delta} = u\partial_2 L(t, u(y \circ \sigma)(t), uy^{\Delta}(t))\,. \quad (7)$$

By assumption $u > 0$ formula (7) is equivalent to

$$\left[\partial_3 L(t, u(y \circ \sigma)(t), uy^{\Delta}(t))\right]^{\Delta} = \partial_2 L(t, u(y \circ \sigma)(t), uy^{\Delta}(t)),$$

and we obtain (4) for $u > 0$.
Similarly, let us take $u < 0$. Then problem (3) reduces to the following nabla problem of the calculus of variations:

$$\begin{aligned} &\text{extremize} \quad \int_a^b uL(t, u(y \circ \rho)(t), uy^{\nabla}(t))\nabla t, \\ &y(a) = \alpha, \;\; y(b) = \beta. \end{aligned} \tag{8}$$

If we set $g(t, y^{\rho}(t), y^{\nabla}(t)) := uL(t, uy^{\rho}(t), uy^{\nabla}(t))$, then problem (8) is equivalent to

$$\begin{aligned} &\text{extremize} \quad \int_a^b g(t, y^{\rho}(t), y^{\nabla}(t))\nabla t, \\ &y(a) = \alpha, \;\; y(b) = \beta. \end{aligned}$$

This is the case of a standard nabla problem of the calculus of variations on time scales [3,28], and we get as necessary optimality condition the nabla differential equation

$$\left[\partial_3 g(t, y^{\rho}(t), y^{\nabla}(t))\right]^{\nabla} = \partial_2 g(t, y^{\nabla}(t), y^{\nabla}(t))$$

that one can write equivalently as

$$\left[\partial_3 L(t, u(y \circ \rho)(t), uy^{\nabla}(t))\right]^{\nabla} = \partial_2 L(t, u(y \circ \rho)(t), uy^{\nabla}(t)),$$

i.e., we obtain (4) for $u < 0$. □

3.3 The general isoperimetric problem

Let $u, w \in \mathbb{R}\backslash\{0\}$ and consider the general isoperimetric problem on time scales. The problem consists of extremizing

$$\mathcal{L}[y] = \int_a^b L(t, (y \circ \xi_u)(t), D_{\uparrow}\overline{y}(t)(u))d_u t \tag{9}$$

subject to the boundary conditions

$$y(a) = \alpha, \quad y(b) = \beta, \tag{10}$$

and the constraint

$$\mathcal{K}[y] = \int_a^b G(t, (y \circ \xi_w)(t), D_{\uparrow}\overline{y}(t)(w))d_w t = K, \tag{11}$$

where α, β, and K are given real numbers.

Definition 35. *We say that y is an* extremal *for $\mathcal{K}$ if y satisfies the equation*

$$\begin{aligned} &D_{\uparrow}\left(\partial_3 G(t, (y \circ \xi_w)(t), D_{\uparrow}\overline{y}(t)(w))\right)(w) \\ &\qquad = w\partial_2 G(t, (y \circ \xi_w)(t), D_{\uparrow}\overline{y}(t)(w)), \quad \forall t \in I_{\kappa^2}^{\kappa^2}, \end{aligned}$$

where $\overline{y}$ *is defined by formula* (1). *An extremizer (i.e., a minimizer or a maximizer) for problem* (9)–(11) *that is not an extremal for* $\mathcal{K}$ *is said to be a normal extremizer; otherwise (i.e., if it is an extremal for* $\mathcal{K}$*), the extremizer is said to be abnormal.*

Proofs of the following two theorems are done following the techniques in the proof of Theorem 34 by considering three cases: (i) u and w are both positive; (ii) u and w are both negative; (iii) u and w are of different signs. Indeed, in case (i) the isoperimetric problem (9)–(11) is reduced to the one studied in [16]; in case (ii) we can apply the results in [2] to obtain Theorem 36 and Theorem 37 in the case $u, w < 0$; when $sign(uw) = -1$ we need to use the necessary optimality conditions for the delta-nabla isoperimetric problems investigated in [17].

Theorem 36. *If* y *is a normal extremizer for the isoperimetric problem* (9)–(11)*, then* y *satisfies the following equation for all* $t \in I_{\kappa^2}^{\kappa^2}$*:*

$$
\begin{aligned}
&D_{\uparrow}(\partial_3 L(t,(y\circ\xi_u)(t), D_{\uparrow}\overline{y}(t)(u)))(u) - u\partial_2 L(t,(y\circ\xi_u)(t), D_{\uparrow}\overline{y}(t)(u)) \\
&= \lambda\Big(D_{\uparrow}(\partial_3 G(t,(y\circ\xi_w)(t), D_{\uparrow}\overline{y}(t)(w)))(w) - w\partial_2 G(t,(y\circ\xi_w)(t), D_{\uparrow}\overline{y}(t)(w))\Big).
\end{aligned}
$$

Next theorem introduces an extra multiplier λ_0, allowing to cover abnormal extremizers.

Theorem 37. *If* y *is an extremizer for the isoperimetric problem* (9)–(11)*, then there exist two constants* λ_0 *and* λ*, not both equal to zero, such that* y *satisfies the following equation for all* $t \in I_{\kappa^2}^{\kappa^2}$*:*

$$
\begin{aligned}
&\lambda_0\Big(D_{\uparrow}(\partial_3 L(t,(y\circ\xi_u)(t), D_{\uparrow}\overline{y}(t)(u)))(u) - u\partial_2 L(t,(y\circ\xi_u)(t), D_{\uparrow}\overline{y}(t)(u))\Big) \\
&= \lambda\Big(D_{\uparrow}(\partial_3 G(t,(y\circ\xi_w)(t), D_{\uparrow}\overline{y}(t)(w)))(w) - w\partial_2 G(t,(y\circ\xi_w)(t), D_{\uparrow}\overline{y}(t)(w))\Big).
\end{aligned}
$$

4 Conclusions

In this paper we claim that the contingent epiderivative, a concept introduced by Aubin with the help of his contingent derivative, is an important tool in the theory of time scales. To show this we propose a new approach to the calculus of variations on time scales, which allows to unify the three different approaches followed so far in the literature: the delta [9, 16, 23]; the nabla [2, 3]; and the delta-nabla [17, 27] approach. Such unification is obtained using, for the first time in the literature of time scales, the contingent epiderivative as the main differentiation tool. The effectiveness of our approach is illustrated by giving general necessary optimality conditions both for the basic and isoperimetric problems of the calculus of variations on time scales. The generalized versions of the Euler-Lagrange conditions are reduced, in particular cases, to recent results obtained in the delta and nabla frameworks.

We trust that the present work opens a new area of research. In particular, it would be interesting to generalize our results for problems of the calculus of

variations on time scales with higher-order contingent epiderivatives, unifying the results of [15] and [28]. In the case of optimal control problems governed by ODE systems (controlled ODEs or differential inclusions) and discrete-time systems there are necessary optimality conditions of different types: mainly using derivative-type constructions (in particular, epi-derivatives) and the coderivative type for the Euler-Lagrange inclusions. These are some interesting questions needing further developments.

Acknowledgments

The authors are supported by the R&D unit *Centre for Research on Optimization and Control* (CEOC) via *The Portuguese Foundation for Science and Technology* (FCT) and the European Community fund FEDER/POCI 2010. Girejko is also supported by the FCT post-doc fellowship SFRH/BPD/48439/2008; Malinowska by Białystok University of Technology, via a project of the Polish Ministry of Science and Higher Education *Wsparcie miedzynarodowej mobilnosci naukowcow*; and Torres by the project UTAustin/MAT/0057/2008.

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